Journal of Applied Mechanics and Technical Physics, Vol. 49, No. 4, pp. 553–567, 2008

ON UNILATERAL CONTACT OF TWO PLATES ALIGNED AT AN ANGLE TO EACH OTHER

A. M. Khludnev

UDC 539.3+517.95

The contact problem for two elastic plates aligned at a prescribed angle to each other is considered. The set of contact points is assumed to be unknown in advance and to be determined only after the problem is solved. Various formulations of the problem are given, and their equivalence is proved. A complete set of boundary conditions fulfilled on the contact domain is found, and the character of satisfaction of these conditions is described. The asymptotic properties of solutions are studied for rigidity parameters of the contacting plates tending to infinity.

Key words: *contact problem, unknown boundary, thin elastic obstacle, crack.*

INTRODUCTION

The variational approach used to describe contact interaction of solids with an unknown contact domain turned out to be very effective. A classical example is the Signorini contact problem for an elastic solid and a rigid solid in the absence of friction. The properties of this problem were studied in [1], which stimulated the research for a wide class of contact problems with an unknown contact domain. Both two-dimensional and three-dimensional contact problems were considered, as well as contact problems for plates and shells (see [2] and the references therein). Equilibrium problems for elastic and inelastic solids containing cracks can also be classified as contact problems if the boundary conditions of mutual non-penetration in the form of a system of equalities and inequalities are imposed on the crack edges [3–5]. These conditions do not allow mutual penetration of the edges; hence, the corresponding mathematical model of the crack is more preferable than the classical model with linear boundary conditions on the crack edges.

Contact problems for bodies of different dimensions with an unknown contact area bear a certain analogy with boundary-value problems of the crack theory, namely, the equilibrium equation for one body is formulated in the domain containing a cut, whereas the boundary conditions on the cut edges are formulated as a system of equalities and inequalities. The character and nature of these boundary conditions, however, differ from the boundary conditions considered in the crack theory. Contact problems are of much interest from the viewpoint of various applications, and their comprehensive mathematical analysis is extremely important.

A contact problem for two elastic plates is considered in the paper, and a full description of the boundary conditions satisfied on the contact set is given. The asymptotic behavior of the solution is studied with variations of model parameters characterizing the rigidity of the contacting bodies. The contact of two plates (upper and lower ones) aligned at a certain angle α to each other is analyzed. Two models are examined (models A and B). The first model implies that the lower plate is deformed in its plane, whereas the lower plate in the second model is subjected to bending only. The equation of equilibrium for the upper plate is written for the domain with the cut. The lower plate can be interpreted as a thin elastic obstacle for the upper plate. The problem of contact

Lavrent'ev Institute of Hydrodynamics, Siberian Division, Russian Academy of Sciences, Novosibirsk 630090; khlud@hydro.nsc.ru. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 49, No. 4, pp. 42–58, July–August, 2008. Original article submitted June 6, 2007.

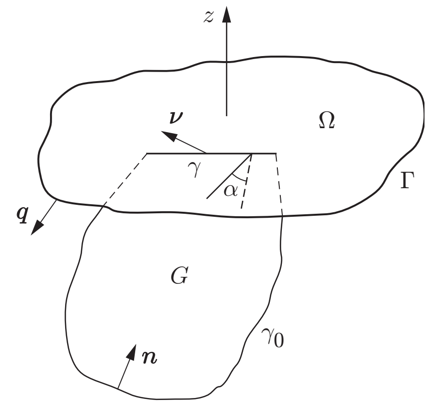

Fig. 1. Problem geometry.

between an elastic plate and an elastic beam was analyzed in a recent paper [6]. Thus, the beam plays the role of a thin elastic obstacle for the plate. It should be noted that unilateral contact problems for plates were extensively analyzed [7–10]. In particular, a thin rigid (non-deformable) obstacle for plates was considered in [10].

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a smooth boundary Γ; the vector of the outward normal to this boundary is denoted by $\boldsymbol{q} = (q_1, q_2)$. We assume that Ω corresponds to the mid-plane of the upper (horizontal) plate. The mid-surface of the lower plate is denoted by G and is assumed to be a bounded domain with a smooth boundary ∂G (see Fig. 1). The angle between Ω and G is denoted by α ($\alpha \in (0, \pi/2]$). We assume that $\Omega \cap G = \emptyset$ and $\Omega \cap \partial G \neq \emptyset$. Let us denote $\gamma_0 = (\partial G) \setminus \Omega$. In this case, we have $\partial G = \gamma \cup \bar{\gamma}_0$. Let $\boldsymbol{\nu} = (\nu_1, \nu_2)$ be the vector of the normal to γ, which is located in the plane Ω. We use $\boldsymbol{n} = (n_1, n_2)$ to denote the unit vector of the inward normal to ∂G, which is located in the plane G. We assume that γ is a connected set (in this particular case, it is an interval) and $\gamma \cap \Gamma = \emptyset$. Let $\Omega_\gamma = \Omega \setminus \bar{\gamma}$.

1. PROBLEM A

1.1. Formulation of Problem A. Let two elastic plates be aligned at an angle α to each other and contact along the line γ in their natural state (see Fig. 1). We assume that the points of the upper plate can move only in the z direction, and the points of the lower plate admit displacements in the mid-plane only. We give the full statement of the problem, which includes several equivalent formulations: differential, variational, and mixed formulations. Let us first consider the differential formulation of the problem. We have to find functions $\boldsymbol{u}(\boldsymbol{x}) = (u_1(\boldsymbol{x}), u_2(\boldsymbol{x}))$, $w(\boldsymbol{y})$, $\boldsymbol{x} = (x_1, x_2) \in G$, and $\boldsymbol{y} = (y_1, y_2) \in \Omega_\gamma$, such that

$$-\operatorname{div}(B\varepsilon(\boldsymbol{u})) = \boldsymbol{g} \quad \text{in} \quad G; \tag{1}$$

$$\Delta^2 w = f \quad \text{in} \quad \Omega_\gamma; \tag{2}$$

$$\boldsymbol{u} = 0 \quad \text{on} \quad \gamma_0; \tag{3}$$

$$w = w_q = 0 \quad \text{on} \quad \Gamma; \tag{4}$$

$$\boldsymbol{un}\sin\alpha + w \geqslant 0, \quad \sigma_n \leqslant 0, \quad \boldsymbol{\sigma}_\tau = 0, \quad \sigma_n(\boldsymbol{un}\sin\alpha + w) = 0 \quad \text{on} \quad \gamma; \tag{5}$$

$$[w] = [w_\nu] = 0, \quad [m(w)] = 0, \quad [t^\nu(w)]\sin\alpha = -\sigma_n \quad \text{on} \quad \gamma. \tag{6}$$

Here $\varepsilon(\boldsymbol{u}) = \{\varepsilon_{ij}(\boldsymbol{u})\}$ and $\sigma = \{\sigma_{ij}\}$ are the strain and stress tensors ($i, j = 1, 2$), respectively,

$$\sigma_n = \sigma_{ij} n_j n_i, \qquad \boldsymbol{\sigma}_\tau = \sigma\boldsymbol{n} - \sigma_n \cdot \boldsymbol{n}, \quad \boldsymbol{\sigma}_\tau = (\sigma_\tau^1, \sigma_\tau^2),$$

$$\sigma\boldsymbol{n} = (\sigma_{1j} n_j, \sigma_{2j} n_j), \qquad \varepsilon_{ij}(\boldsymbol{u}) = (u_{i,j} + u_{j,i})/2, \qquad i, j = 1, 2,$$

$B = \{b_{ijkl}\}$ $(i, j, k, l = 1, 2)$ is the tensor of elasticity moduli, $b_{ijkl} \in L^\infty(G)$:

$$b_{ijkl} = b_{jikl} = b_{ijlk}, \qquad b_{ijkl}\xi_{kl}\xi_{ij} \geqslant c|\xi|^2, \qquad c > 0,$$

$[v] = v^+ - v^-$ is the jump of the function v on γ; the quantities $v^\pm$ refer to the positive and negative (with respect to the normal $\boldsymbol{\nu}$) edges of the cut $\gamma^\pm$. All functions with two subscripts are assumed to be symmetric over these subscripts, i.e., $\xi_{ij} = \xi_{ji}$, etc. Summation is performed over repeated subscripts, and the functions $\boldsymbol{g} = (g_1, g_2) \in L^2(G)$ and $f \in L^2(\Omega)$ are given. In addition,

$$w_\nu = \frac{\partial w}{\partial \nu}, \qquad w_q = \frac{\partial w}{\partial q}, \qquad m(w) = \varkappa_1 \Delta w + (1 - \varkappa_1)\,\frac{\partial^2 w}{\partial \nu^2},$$

$$t^\nu(w) = \frac{\partial}{\partial \nu}\Big(\Delta w + (1 - \varkappa_1)\frac{\partial^2 w}{\partial s^2}\Big), \qquad (s_1, s_2) = (-\nu_2, \nu_1),$$

where $\varkappa_1$ is Poisson's ratio for the upper plate; $m(w)$ and $t^\nu(w)$ are the bending moment and the shear force for the upper plate.

It should be noted that Eqs. (1) and (2) are equilibrium equations, and $\sigma = B\varepsilon(\boldsymbol{u})$ is Hooke's law $[\sigma = \sigma(\boldsymbol{u})]$. Relations (3) and (4) provide a clamped state of the plates on γ_0 and Γ, respectively. The first inequality in (5) describes mutual non-penetration of the plates. The equilibrium equation (2) is valid in the domain Ω_γ with the cut (crack) γ, and the boundary conditions (5) and (6) are formulated as a system of equalities and inequalities.

Let us consider the variational formulation of problem (1)–(6), which implies, in particular, the solution existence. The differential formulation of this problem is equivalent to the variational formulation.

We consider the Sobolev spaces

$$H^1_{\gamma_0}(G) = \{v \in H^1(G)\colon v = 0 \text{ in } \gamma_0\}, \qquad H^2_0(\Omega) = \{v \in H^2(\Omega)\colon v = v_q = 0 \text{ in } \Gamma\}$$

and the bilinear form

$$a_\Omega(w, \bar w) = \int\limits_\Omega (w_{,11}\bar w_{,11} + w_{,22}\bar w_{,22} + \varkappa_1(w_{,11}\bar w_{,22} + w_{,22}\bar w_{,11}) + 2(1 - \varkappa_1)w_{,12}\bar w_{,12}).$$

Let $(u, v)_\Omega$ denote the scalar product in $L^2(\Omega)$, i.e., $(u, v)_\Omega = \int\limits_\Omega uv$. We denote

$$K = \{(\boldsymbol{u}, w)\colon \boldsymbol{u} = (u_1, u_2) \in H^1_{\gamma_0}(G),\ w \in H^2_0(\Omega),\ \boldsymbol{un}\sin\alpha + w \geqslant 0 \text{ in } \gamma\}$$

and consider the energy functional

$$E(\boldsymbol{u}, w) = (\sigma(\boldsymbol{u}), \varepsilon(\boldsymbol{u}))_G/2 - (\boldsymbol{g}, \boldsymbol{u})_G + a_\Omega(w, w)/2 - (f, w)_\Omega.$$

We can find the solution of the minimization problem

$$\inf_{(\boldsymbol{u}, w) \in K} E(\boldsymbol{u}, w), \tag{7}$$

which is equivalent to the variational inequality

$$(\boldsymbol{u}, w) \in K; \tag{8}$$

$$(\sigma(\boldsymbol{u}), \varepsilon(\bar{\boldsymbol{u}} - \boldsymbol{u}))_G - (\boldsymbol{g}, \bar{\boldsymbol{u}} - \boldsymbol{u})_G + a_\Omega(w, \bar w - w) - (f, \bar w - w)_\Omega \geqslant 0 \quad \forall\,(\bar{\boldsymbol{u}}, \bar w) \in K. \tag{9}$$

Note that the functional E is coercive and weakly lower semi-continuous on the space $[H^1_{\gamma_0}(G)]^2 \times H^2_0(\Omega)$. Moreover, the set K is weakly closed. Hence, the minimization problem (7) has a (unique) solution that satisfies the variational inequality (8), (9).

Let us prove that problems (1)–(6) and (8), (9) are equivalent.

First, we obtain relations (1)–(6) from problem (8), (9) and find in which sense the boundary conditions (5) and (6) are satisfied. Note that Eqs. (1) and (2) follow from Eq. (9) and are satisfied in the sense of distributions. Indeed, substituting the test functions $(\bar{\boldsymbol{u}}, \bar w) = (\boldsymbol{u} \pm \boldsymbol{\psi}, w \pm \varphi)$, $\boldsymbol{\psi} = (\psi_1, \psi_2) \in C_0^\infty(G)$, and $\varphi \in C_0^\infty(\Omega_\gamma)$ into Eq. (9), we obtain (1) and (2).

We choose $(\bar{\boldsymbol{u}}, \bar w) = (\boldsymbol{u} + \boldsymbol{\psi}, w)$ as test functions in (9). Here $\boldsymbol{\psi} = (\psi_1, \psi_2) \in H^1_{\gamma_0}(G)$ and $\boldsymbol{\psi}_n = \boldsymbol{\psi} n \geqslant 0$ on γ. As a result, we obtain

$$(\sigma(\boldsymbol{u}), \varepsilon(\boldsymbol{\psi}))_G - (\boldsymbol{g}, \boldsymbol{\psi})_G \geqslant 0. \tag{10}$$

The following Green's formula is valid [4, 11]:

$$(\sigma(\boldsymbol{u}), \varepsilon(\boldsymbol{\psi}))_G = -(\operatorname{div} \sigma(\boldsymbol{u}), \boldsymbol{\psi})_G - \langle \sigma_n, \boldsymbol{\psi}_n \rangle_{1/2,\partial G} - \langle \sigma_\tau, \boldsymbol{\psi}_\tau \rangle_{1/2,\partial G}. \tag{11}$$

Here the notation $\langle \cdot, \cdot \rangle_{1/2,\partial G}$ indicates the duality pairing between $H^{-1/2}(\partial G)$ and $H^{1/2}(\partial G)$, where the space $H^{-1/2}(\partial G)$ is dual of $H^{1/2}(\partial G)$; $\boldsymbol{\psi} = \boldsymbol{\psi}_n \boldsymbol{n} + \boldsymbol{\psi}_\tau$. Taking into account the equilibrium equations

$$-\operatorname{div} \sigma(\boldsymbol{u}) = \boldsymbol{g} \qquad \text{in} \quad G$$

we use Eq. (10) to obtain

$$-\langle \sigma_n, \boldsymbol{\psi}_n \rangle_{1/2,\partial G} - \langle \boldsymbol{\sigma}_\tau, \boldsymbol{\psi}_\tau \rangle_{1/2,\partial G} \geqslant 0. \tag{12}$$

As the functions $\boldsymbol{\psi}_\tau$ are arbitrary on ∂G, inequality (12) implies the relation

$$\langle \boldsymbol{\sigma}_\tau, \boldsymbol{\psi}_\tau \rangle_{1/2,\partial G} = 0. \tag{13}$$

We consider the space $H^{1/2}_{00}(\gamma)$ where the norm is determined as follows (see [4]):

$$\|v\|^2_{H^{1/2}_{00}(\gamma)} = \|v\|^2_{H^{1/2}(\gamma)} + \int\limits_\gamma \rho^{-1} v^2.$$

Here $\rho(\boldsymbol{y}) = \operatorname{dist}(\boldsymbol{y}, \partial\gamma)$. We also introduce the space

$$H^{3/2}_{00}(\gamma) = \Big\{ v \in H^{3/2}_0(\gamma)\colon \quad \int\limits_\gamma \frac{|\nabla v|^2}{\rho} < \infty \Big\}$$

and use the following statement. Let the function u be defined on γ. We use $\bar{u}$ to denote the extension of u by the zero outside γ, i.e.,

$$\bar{u} = \begin{cases} u & \text{in} \quad \gamma, \\ 0 & \text{in} \quad \partial G \setminus \gamma. \end{cases}$$

Then $\bar{u} \in H^{i/2}(\partial G)$ if and only if $u \in H^{i/2}_{00}(\gamma)$, $i = 1, 3$ (see [4, 11]). By virtue of this property and the equality $\boldsymbol{\psi} = 0$ on γ_0, relation (13) can be written as

$$\langle \boldsymbol{\sigma}_\tau, \boldsymbol{\psi}_\tau \rangle^{00}_{1/2,\gamma} = 0, \tag{14}$$

where the notation $\langle \cdot, \cdot \rangle^{00}_{1/2,\gamma}$ means the duality pairing between $H^{1/2}_{00}(\gamma)$ and the dual space $H^{-1/2}_{00}(\gamma)$. Relation (14) yields the equality

$$\boldsymbol{\sigma}_\tau = (\sigma^1_\tau, \sigma^2_\tau) = 0 \quad \text{in the sense} \quad H^{-1/2}_{00}(\gamma), \tag{15}$$

and inequality (12) yields the inequality

$$\sigma_n \leqslant 0 \quad \text{in the sense} \quad H^{-1/2}_{00}(\gamma). \tag{16}$$

We consider the extension of γ in the domain Ω up to a closed curve Σ of class $C^{1,1}$, such that $\Sigma \subset \Omega$. In this case, the domain Ω is divided into two subdomains Ω_1 and Ω_2 with the boundaries Σ and $\Sigma \cup \Gamma$, respectively. We assume that the normal $\boldsymbol{\nu}$ is defined on Σ, being an outward normal to Ω_1. We choose $(\bar{\boldsymbol{u}}, \bar{w}) = (\boldsymbol{u}, w + \varphi)$ as test functions in (9). Here $\varphi \geqslant 0$ on γ, $\varphi \in H^2_0(\Omega)$. As a result, we obtain the relation

$$a_\Omega(w, \varphi) - (f, \varphi)_\Omega \geqslant 0. \tag{17}$$

We consider the space

$$V = \{ v \in H^2(\Omega_1)\colon \quad \Delta^2 v \in L^2(\Omega_1) \}.$$

For $v \in V$, we can determine $m(v) \in H^{-1/2}(\Sigma)$ and $t^\nu(v) \in H^{-3/2}(\Sigma)$. Then, the following Green's function is valid [4, 12]:

$$(\varphi, \Delta^2 v)_{\Omega_1} = a_{\Omega_1}(\varphi, v) + \langle t^\nu(v), \varphi \rangle_{3/2,\Sigma} - \langle m(v), \varphi_\nu \rangle_{1/2,\Sigma} \quad \forall\, \varphi \in H^2(\Omega_1). \tag{18}$$

Here the notation $\langle\,\cdot\,,\,\cdot\,\rangle_{i/2,\Sigma}$ means the duality pairing between the space $H^{-i/2}(\Sigma)$ and the dual space $H^{i/2}(\Sigma)$, $i = 1, 3$. Green's formula allows us to derive the following inequality from Eqs. (17) and (2):

$$-\langle [m(w)], \varphi_\nu \rangle_{1/2,\Sigma} + \langle [t^\nu(w)], \varphi \rangle_{3/2,\Sigma} \geqslant 0.$$

As φ_ν are arbitrary functions on Σ, we obtain

$$[m(w)] = 0 \quad \text{in the sense} \quad H^{-1/2}(\Sigma); \tag{19}$$

$$\langle [t^\nu(w)], \varphi \rangle_{3/2,\Sigma} \geqslant 0 \quad \forall\, \varphi \in H_0^2(\Omega), \qquad \varphi \geqslant 0 \quad \text{on} \quad \gamma. \tag{20}$$

We substitute $(\bar{\boldsymbol{u}}, \bar{w}) = (\boldsymbol{u} \pm \boldsymbol{\psi}, w \pm \varphi)$ as test functions in (9), and $\boldsymbol{\psi}_n \sin\alpha = -\varphi$ on γ, $\boldsymbol{\psi} = (\psi_1, \psi_2) \in H^1_{\gamma_0}(G)$, and $\varphi \in H_0^2(\Omega)$. In this case, $\boldsymbol{\psi}_n \in H^{1/2}_{00}(\gamma)$. In addition, we assume that $\varphi = 0$ on $\Sigma \setminus \gamma$. Then, $\varphi \in H^{3/2}_{00}(\gamma)$. This substitution yields

$$(\sigma(\boldsymbol{u}), \varepsilon(\boldsymbol{\psi}))_G - (\boldsymbol{g}, \boldsymbol{\psi})_G + a_\Omega(w, \varphi) - (f, \varphi)_\Omega = 0. \tag{21}$$

By virtue of Eqs. (1), (2), (15), and (19) and with the use of Green's formulas (11) and (18), Eq. (21) yields

$$\langle [t^\nu(w)], \varphi \rangle_{3/2,\Sigma} - \langle \sigma_n, \boldsymbol{\psi}_n \rangle^{00}_{1/2,\gamma} = 0.$$

As $\varphi = 0$ on $\Sigma \setminus \gamma$, the latter relation can be written in the form

$$\langle [t^\nu(w)], \varphi \rangle^{00}_{3/2,\gamma} - \langle \sigma_n, \boldsymbol{\psi}_n \rangle^{00}_{1/2,\gamma} = 0. \tag{22}$$

In our case, however, $\langle \sigma_n, \boldsymbol{\psi}_n \rangle^{00}_{3/2,\gamma} = \langle \sigma_n, \boldsymbol{\psi}_n \rangle^{00}_{1/2,\gamma}$; hence, Eq. (22) yields

$$[t^\nu(w)] \sin\alpha = -\sigma_n \quad \text{in the sense} \quad H^{-1/2}_{00}(\gamma). \tag{23}$$

We choose $(\bar{\boldsymbol{u}}, \bar{w}) = (\boldsymbol{u} + \boldsymbol{\psi}, w + \varphi)$ as a test function in (9), with $\boldsymbol{\psi}_n \sin\alpha + \varphi \geqslant 0$ on γ, $\boldsymbol{\psi} = (\psi_1, \psi_2) \in H^1_{\gamma_0}(G)$, and $\varphi \in H_0^2(\Omega)$. As a result, we obtain

$$(\sigma(\boldsymbol{u}), \varepsilon(\boldsymbol{\psi}))_G - (\boldsymbol{g}, \boldsymbol{\psi})_G + a_\Omega(w, \varphi) - (f, \varphi)_\Omega \geqslant 0.$$

Applying Green's formulas (11) and (18) to this inequality, by virtue of Eqs. (1), (2), and (15)–(19), we obtain

$$\langle [t^\nu(w)], \varphi \rangle_{3/2,\Sigma} - \langle \sigma_n, \boldsymbol{\psi}_n \rangle^{00}_{1/2,\gamma} \geqslant 0 \quad \forall\, (\psi, \varphi) \in K. \tag{24}$$

Inequality (24) ensures the exact formulation of the relations [see (5) and (6)]

$$\sigma_n \leqslant 0, \qquad [t^\nu(w)] \sin\alpha = -\sigma_n \quad \text{on} \quad \gamma.$$

It is also worth noting that relations (16), (20), and (23) follow from inequality (24).

Choosing $(\bar{\boldsymbol{u}}, \bar{w}) = (0, 0)$ and $(\bar{\boldsymbol{u}}, \bar{w}) = 2(\boldsymbol{u}, w)$ in (9), we obtain the relation

$$\langle [t^\nu(w)], w \rangle_{3/2,\Sigma} - \langle \sigma_n, \boldsymbol{u}_n \rangle^{00}_{1/2,\gamma} = 0,$$

which is the exact formulation of the last relations in Eqs. (5) and (6).

The above-discussed situation implies that the first term in (24) is independent of the choice of Σ. It is only important that the curve Σ satisfies the indicated conditions of smoothness. Moreover, the first term in (24) is independent of the values of φ on $\Sigma \setminus \bar{\gamma}$. In other words, if $\varphi^1 = \varphi^2$ on γ, then we have

$$\langle [t^\nu(w)], \varphi^1 \rangle_{3/2,\Sigma} = \langle [t^\nu(w)], \varphi^2 \rangle_{3/2,\Sigma}.$$

The system of the boundary conditions (3)–(6) is complete; in particular, the variational inequality (8), (9) can be derived from Eqs. (1)–(6).

Let us discuss the so-called mixed formulation of problem (1)–(6). In contrast to the differential and variational formulations, the mixed formulation contains a set of admissible stresses and moments. The plate displacements are formally found from rather wide classes of functions, which do not allow any boundary conditions to be discussed because of insufficient smoothness of the boundary. Nevertheless, the proposed formulation of the problem contains all necessary information on displacements. We use the relation between $m = \{m_{ij}\}$ and $\nabla\nabla w = \{w_{,ij}\}$ in the form $m = D\nabla\nabla w, D = \{d_{ijkl}\}$, and $d_{ijkl} = d_{jikl} = d_{klij}$ $(i, j, k, l = 1, 2)$. Further, we will need a particular form of D corresponding to the relations

$$m_{11} = w_{,11} + \varkappa_1 w_{,22}, \qquad m_{22} = w_{,22} + \varkappa_1 w_{,11}, \qquad m_{12} = (1 - \varkappa_1) w_{,12}.$$

We write problem (1)–(6) in the following equivalent form. We have to find functions $\boldsymbol{u}(\boldsymbol{x}) = (u_1(\boldsymbol{x}), u_2(\boldsymbol{x}))$, $\sigma(\boldsymbol{x}) = \{\sigma_{ij}(\boldsymbol{x})\}$, $w(\boldsymbol{y})$, and $m(\boldsymbol{y}) = \{m_{ij}(\boldsymbol{y})\}$ ($i, j = 1, 2$; $\boldsymbol{x} \in G$ and $\boldsymbol{y} \in \Omega_\gamma$) such that

$$-\operatorname{div}\sigma = \boldsymbol{g} \quad \text{in} \quad G; \tag{25}$$

$$B^{-1}\sigma = \varepsilon(\boldsymbol{u}) \quad \text{in} \quad G; \tag{26}$$

$$\nabla\nabla m = f \quad \text{in} \quad \Omega_\gamma; \tag{27}$$

$$D^{-1}m = \nabla\nabla w \quad \text{in} \quad \Omega_\gamma, \tag{28}$$

$$\boldsymbol{u} = 0 \quad \text{on} \quad \gamma_0; \tag{29}$$

$$w = w_q = 0 \quad \text{on} \quad \Gamma; \tag{30}$$

$$\boldsymbol{un}\sin\alpha + w \geqslant 0, \quad \sigma_n \leqslant 0, \quad \boldsymbol{\sigma}_\tau = 0, \quad \sigma_n(\boldsymbol{un}\sin\alpha + w) = 0 \quad \text{on} \quad \gamma; \tag{31}$$

$$[w] = [w_\nu] = 0, \quad [m_\nu] = 0, \quad [T^\nu(m)]\sin\alpha = -\sigma_n \quad \text{on} \quad \gamma. \tag{32}$$

Here

$$\nabla\nabla m = m_{ij,ij}, \qquad m_\nu = m_{ij}\nu_j\nu_i,$$

$$T^\nu(m) = m_{ij,k}s_k s_j \nu_i + m_{ij,j}\nu_i, \qquad (s_1, s_2) = (-\nu_2, \nu_1).$$

The tensor B^{-1} is obtained by inversion of Hooke's law $\sigma = B\varepsilon(\boldsymbol{u})$, and the tensor D^{-1} is obtained by inversion of the law $m = D\nabla\nabla w$.

In addition to Eq. (22), we need the following variant of Green's formula (see [4]). If $m = \{m_{ij}\}$ ($i, j = 1, 2$) and $m \in L^2(\Omega_1)$ and $\nabla\nabla m \in L^2(\Omega_1)$, we can determine $m_\nu \in H^{-1/2}(\Sigma)$ and $T^\nu(m) \in H^{-3/2}(\Sigma)$ with

$$(\varphi, m_{ij,ij})_{\Omega_1} = (\varphi_{,ij}, m_{ij})_{\Omega_1} + \langle T^\nu(m), \varphi\rangle_{3/2,\Sigma} - \langle m_\nu, \varphi_\nu\rangle_{1/2,\Sigma} \qquad \forall\, \varphi \in H^2(\Omega_1).$$

We introduce a set of admissible stresses and moments

$$L = \{(\bar\sigma, \bar m)\colon \quad \bar\sigma, \operatorname{div}\bar\sigma \in L^2(G), \quad \bar m, \nabla\nabla\bar m \in L^2(\Omega_\gamma),$$

$$\bar\sigma_n \leqslant 0, \quad \bar{\boldsymbol{\sigma}}_\tau = 0, \quad [\bar m_\nu] = 0, \quad [T^\nu(\bar m)]\sin\alpha = -\bar\sigma_n \text{ on } \gamma\}.$$

Here $\bar\sigma = \{\bar\sigma_{ij}\}$ and $\bar m = \{\bar m_{ij}\}$, $i, j = 1, 2$; the boundary conditions for $\bar\sigma$ and $\bar m$ in determining L are satisfied in the following sense:

$$\bar{\boldsymbol{\sigma}}_\tau = (\bar\sigma_\tau^1, \bar\sigma_\tau^2) = 0 \quad \text{in the sense} \quad H_{00}^{-1/2}(\gamma),$$

$$[\bar m_\nu] = 0 \quad \text{in the sense} \quad H_{00}^{-1/2}(\Sigma).$$

The inequality $\bar\sigma_n \leqslant 0$ and the equality $[T^\nu(\bar m)]\sin\alpha = -\bar\sigma_n$ are satisfied in the sense

$$\langle [T^\nu(\bar m)], \bar w\rangle_{3/2,\Sigma} - \langle \bar\sigma_n, \bar{\boldsymbol{u}}_n\rangle_{1/2,\gamma}^{00} \geqslant 0 \qquad \forall\, (\bar{\boldsymbol{u}}, \bar w) \in K.$$

We multiply Eqs. (26) and (28) by $\bar\sigma - \sigma$ and $\bar m - m$, respectively, integrate them with respect to G and Ω_γ, and summarize. Here $(\bar\sigma, \bar m) \in L$. As a result, we obtain the following formulation of the problem. We have to find functions $\boldsymbol{u}(\boldsymbol{x}) = (u_1(\boldsymbol{x}), u_2(\boldsymbol{x}))$, $\sigma(\boldsymbol{x}) = \{\sigma_{ij}(\boldsymbol{x})\}$, $w(\boldsymbol{y})$, and $m(\boldsymbol{y}) = \{m_{ij}(\boldsymbol{y})\}$ ($i, j = 1, 2$; $\boldsymbol{x} \in G$ and $\boldsymbol{y} \in \Omega_\gamma$) such that

$$\boldsymbol{u} \in L^2(G), \qquad w \in L^2(\Omega_\gamma), \qquad (\sigma, m) \in L; \tag{33}$$

$$-\operatorname{div}\sigma = \boldsymbol{g} \quad \text{in} \quad G; \tag{34}$$

$$\nabla\nabla m = f \quad \text{in} \quad \Omega_\gamma; \tag{35}$$

$$(B^{-1}\sigma, \bar\sigma - \sigma)_G + (\boldsymbol{u}, \operatorname{div}\bar\sigma - \operatorname{div}\sigma)_G + (D^{-1}m, \bar m - m)_{\Omega_\gamma}$$

$$-(w, \nabla\nabla\bar m - \nabla\nabla m)_{\Omega_\gamma} \geqslant 0 \qquad \forall\, (\bar\sigma, \bar m) \in L. \tag{36}$$

Relations (33)–(36) represent the mixed formulation of problem (1)–(6). It should be noted that the displacements $\boldsymbol{u}$ and w in this formulation are sought in the spaces L^2; therefore, problem (33)–(36) does not contain any boundary conditions for displacements. It should be also noted that problem (1)–(6) is equivalent to problem (33)–(36). To prove this fact, it suffices to derive (1)–(6) from (33)–(36) (see below). Equation (36) implies that the following equations are valid in the sense of distributions:

$$B^{-1}\sigma = \varepsilon(\boldsymbol{u}) \quad \text{in} \quad G, \qquad D^{-1}m = \nabla\nabla w \quad \text{in} \quad \Omega_\gamma. \tag{37}$$

Thus, by virtue of Eq. (33), we obtain inclusions $\boldsymbol{u} = (u_1, u_2) \in H^1(G)$ and $w \in H^2(\Omega_\gamma)$. This means that the functions $\boldsymbol{u}$ and w are actually more smooth than in Eq. (33), and we can speak about the boundary conditions for displacements.

From Eqs. (33)–(36), we can derive the boundary conditions (4). We show that

$$[w] = [w_\nu] = 0 \quad \text{on} \quad \gamma. \tag{38}$$

For this purpose, we find the solution $\tilde{w}$ of the problem

$$\Delta^2 \tilde{w} = f \quad \text{in} \quad \Omega_\gamma; \tag{39}$$

$$\tilde{w} = \tilde{w}_q = 0 \quad \text{on} \quad \Gamma; \tag{40}$$

$$m(\tilde{w}) = \varphi, \quad t^\nu(\tilde{w}) = \xi \quad \text{on} \quad \gamma^\pm, \tag{41}$$

where φ and ξ are arbitrary functions in $L^2(\gamma)$. Problem (39)–(41) admits a variational formulation. We have to fund a function $\tilde{w}$, such that

$$\tilde{w} \in H^2_\Gamma(\Omega_\gamma); \tag{42}$$

$$a_{\Omega_\gamma}(\tilde{w}, v) - (f, v)_{\Omega_\gamma} - (\xi, [v])_\gamma + (\varphi, [v_\nu])_\gamma = 0 \quad \forall\, v \in H^2_\Gamma(\Omega_\gamma), \tag{43}$$

where

$$H^2_\Gamma(\Omega_\gamma) = \{v \in H^2(\Omega_\gamma)\colon \quad v = v_q = 0 \text{ on } \Gamma\}.$$

The solution $\tilde{w}$ of problem (42), (43) satisfies the conditions

$$[m(\tilde{w})] = 0 \quad \text{in the sense} \quad H^{-1/2}(\Sigma),$$

$$[t^\nu(\tilde{w})] = 0 \quad \text{in the sense} \quad H^{-3/2}(\Sigma).$$

By choosing the test functions in Eq. (36) in the form $(\bar{\sigma}, \bar{m}) = (\sigma, m) \pm (0, \tilde{m})$, $\tilde{m} = \{\tilde{m}_{ij}\}$ $(i, j = 1, 2)$, and $\tilde{m} = D\nabla\nabla\tilde{w}$, we obtain the relation

$$(D^{-1}m, \tilde{m})_{\Omega_\gamma} - (w, \nabla\nabla\tilde{m})_{\Omega_\gamma} = 0,$$

which implies, by virtue of Eq. (37), that

$$\langle T^\nu(\tilde{m}), [w]\rangle_{3/2,\Sigma} - \langle \tilde{m}_\nu, [w_\nu]\rangle_{1/2,\Sigma} = 0.$$

Here $T^\nu(\tilde{m}) = t^\nu(\tilde{w})$ and $\tilde{m}_\nu = m(\tilde{w})$ (by virtue of the previous comments, the jumps of these quantities on Σ are equal to zero). Equations (42) and (43) yield the relation

$$\langle T^\nu(\tilde{m}), [v]\rangle_{3/2,\Sigma} - \langle \tilde{m}_\nu, [v_\nu]\rangle_{1/2,\Sigma} = (\xi, [v])_\gamma - (\varphi, [v_\nu])_\gamma \quad \forall\, v \in H^2_\Gamma(\Omega_\gamma).$$

Hence, we have

$$(\xi, [w])_\gamma - (\varphi, [w_\nu])_\gamma = 0,$$

and the boundary conditions (38) are satisfied owing to the arbitrariness of φ and ξ. In particular, we obtain $w \in H^2_0(\Omega)$.

Let us prove that the function $\boldsymbol{u}$ in (33)–(36) satisfies the condition

$$\boldsymbol{u} = 0 \quad \text{on} \quad \gamma_0. \tag{44}$$

We recall that $\boldsymbol{u} = (u_1, u_2) \in H^1(G)$. We divide γ_0 into two parts: $\gamma_0 = \gamma_1 \cup \gamma_2$, where γ_i $(i = 1, 2)$ are smooth curves and introduce the notation

$$H^1_{\gamma_1}(G) = \{v \in H^1(G)\colon v = 0 \text{ on } \gamma_1\}.$$

Let $\boldsymbol{\xi} = (\xi_1, \xi_2) \in L^2(\gamma_2)$ be an arbitrary function. There exists a solution of the problem

$$\tilde{\boldsymbol{u}} = (\tilde{u}_1, \tilde{u}_2) \in H^1_{\gamma_1}(G); \tag{45}$$

$$(\sigma(\tilde{\boldsymbol{u}}), \varepsilon(\boldsymbol{v}))_G - (\boldsymbol{g}, \boldsymbol{v})_G + (\boldsymbol{\xi}, \boldsymbol{v})_{\gamma_2} = 0 \quad \forall\, \boldsymbol{v} = (v_1, v_2) \in H^1_{\gamma_1}(G), \tag{46}$$

where $\sigma(\tilde{\boldsymbol{u}}) = B\varepsilon(\tilde{\boldsymbol{u}})$. Obviously, this solution satisfies the relations

$$-\operatorname{div}(B\varepsilon(\tilde{\boldsymbol{u}})) = \boldsymbol{g} \quad \text{in} \quad G,$$

$$\tilde{\boldsymbol{u}} = 0 \quad \text{on} \quad \gamma_1,$$

$$\sigma(\tilde{\boldsymbol{u}})\boldsymbol{n} = 0 \quad \text{on} \quad \gamma,$$

$$\sigma(\tilde{\boldsymbol{u}})\boldsymbol{n} = \boldsymbol{\xi} \quad \text{on} \quad \gamma_2.$$

We denote $\tilde{\sigma} = \sigma(\tilde{\boldsymbol{u}})$ and choose a cut-off function η, $\eta = 1$ in a small vicinity of a fixed point on γ_2. In this case, we have $\pm(\eta\tilde{\sigma}, 0) \in L$. We choose $(\bar{\sigma}, \bar{m}) = (\sigma, m) \pm (\eta\tilde{\sigma}, 0)$ as a test function in Eq. (36). Then, we obtain

$$(B^{-1}\sigma, \eta\tilde{\sigma})_G + (\boldsymbol{u}, \operatorname{div}(\eta\tilde{\sigma}))_G = 0$$

and, hence, by virtue of Eq. (37),

$$\langle (\eta\tilde{\sigma})\boldsymbol{n}, \boldsymbol{u}\rangle_{1/2, \partial G} = 0.$$

This relation can be written as

$$\langle \tilde{\sigma}\boldsymbol{n}, \eta\boldsymbol{u}\rangle_{1/2, \partial G} = 0. \tag{47}$$

Simultaneously, identity (46) yields the relation

$$\langle \tilde{\sigma}\boldsymbol{n}, \boldsymbol{v}\rangle_{1/2, \partial G} = (\boldsymbol{\xi}, \boldsymbol{v})_{\gamma_2} \qquad \forall\, \boldsymbol{v} = (v_1, v_2) \in H^1_{\gamma_1}(G). \tag{48}$$

As $\eta\boldsymbol{u} = (\eta u_1, \eta u_2) \in H^1_{\gamma_1}(G)$, we find the following relation from Eqs. (47) and (48):

$$(\boldsymbol{\xi}, \eta\boldsymbol{u})_{\gamma_2} = 0.$$

By virtue of the arbitrariness of $\boldsymbol{\xi}$, the equality $\eta\boldsymbol{u} = 0$ is satisfied on γ_2, which yields the necessary boundary condition (44).

Let us prove that the solution of problem (33)–(36) satisfies the boundary condition

$$\boldsymbol{un}\sin\alpha + w \geqslant 0 \quad \text{on} \quad \gamma. \tag{49}$$

We consider the solution $\tilde{w}$ of the problem

$$\tilde{w} \in H^2_\Gamma(\Omega_\gamma); \tag{50}$$

$$a_{\Omega_\gamma}(\tilde{w}, v) - (f, v)_{\Omega_\gamma} - (\varphi, v)_{\gamma^+} = 0 \quad \forall\, v \in H^2_\Gamma(\Omega_\gamma), \tag{51}$$

where $\varphi \in L^2(\gamma)$ is an arbitrary function ($\varphi \geqslant 0$). The subscript γ^+ means that we use the trace of the function v on the edge γ^+. The solution of this problem satisfies the relations

$$\Delta^2\tilde{w} = f \quad \text{in} \quad \Omega_\gamma,$$

$$\tilde{w} = \tilde{w}_q = 0 \quad \text{on} \quad \Gamma,$$

$$m(\tilde{w}) = 0 \quad \text{on} \quad \gamma^\pm,$$

$$t^\nu(\tilde{w}) = \varphi \quad \text{on} \quad \gamma^+, \qquad t^\nu(\tilde{w}) = 0 \quad \text{on} \quad \gamma^-.$$

We also find the solution $\tilde{u}$ of the problem

$$\tilde{\boldsymbol{u}} = (\tilde{u}_1, \tilde{u}_2) \in H^1_{\gamma_0}(G); \tag{52}$$

$$(\sigma(\tilde{\boldsymbol{u}}), \varepsilon(\boldsymbol{v}))_G - (\boldsymbol{g}, \boldsymbol{v})_G - (\varphi\sin\alpha, v_n)_\gamma = 0 \quad \forall\, \boldsymbol{v} = (v_1, v_2) \in H^1_{\gamma_0}(G), \tag{53}$$

where $\boldsymbol{v}_n = \boldsymbol{vn}$ and $\sigma(\tilde{\boldsymbol{u}}) = B\varepsilon(\tilde{\boldsymbol{u}})$. Obviously, $\tilde{\boldsymbol{u}}$ satisfies the equations and boundary conditions

$$-\operatorname{div}(B\varepsilon(\tilde{\boldsymbol{u}})) = \boldsymbol{g} \quad \text{in} \quad G,$$

$$\tilde{\boldsymbol{u}} = 0 \quad \text{on} \quad \gamma_0,$$

$$\sigma_n(\tilde{\boldsymbol{u}}) = -\varphi \sin\alpha, \qquad \boldsymbol{\sigma}_\tau(\tilde{\boldsymbol{u}}) = 0 \quad \text{on} \quad \gamma.$$

We determine the tensors $\tilde{\sigma} = \sigma(\tilde{\boldsymbol{u}})$, $\tilde{m} = \{\tilde{m}_{ij}\}$ $(i, j = 1, 2)$, and $\tilde{m} = D\nabla\nabla\tilde{w}$. Then the function $(\bar{\sigma}, \bar{m}) = (\sigma, m) + (\tilde{\sigma}, \tilde{m})$ can be chosen in Eq. (36) as a test function. Indeed, we have $\tilde{\sigma}_n \leqslant 0$, $\tilde{\boldsymbol{\sigma}}_\tau = 0$, $[\tilde{m}_\nu] = 0$ on γ. Moreover, identity (51) implies that

$$\langle [T^\nu(\tilde{m})], \bar{w}\rangle_{3/2,\Sigma} = (\varphi, \bar{w})_\gamma \qquad \forall\, \bar{w} \in H_0^2(\Omega),$$

and Eq. (53) yields

$$-\langle \tilde{\sigma}_n, \bar{\boldsymbol{u}}_n\rangle^{00}_{1/2,\gamma} = (\varphi, \bar{\boldsymbol{u}}_n \sin\alpha)_\gamma \qquad \forall\, \bar{\boldsymbol{u}} = (\bar{u}_1, \bar{u}_2) \in H^1_{\gamma_0}(G).$$

Summarizing two last relations, we obtain the equality

$$\langle [T^\nu(\tilde{m})], \bar{w}\rangle_{3/2,\Sigma} - \langle \tilde{\sigma}_n, \bar{\boldsymbol{u}}_n\rangle^{00}_{1/2,\gamma} = (\varphi, \bar{w} + \bar{\boldsymbol{u}}_n \sin\alpha)_\gamma. \tag{54}$$

If $\bar{w} + \bar{\boldsymbol{u}}_n \sin\alpha \geqslant 0$ on γ, i.e., $(\bar{\boldsymbol{u}}, \bar{w}) \in K$, then the right side of Eq. (54) is non-negative and $(\tilde{\sigma}, \tilde{m}) \in L$; hence, $(\bar{\sigma}, \bar{m}) = (\sigma, m) + (\tilde{\sigma}, \tilde{m}) \in L$. Therefore, substituting $(\bar{\sigma}, \bar{m})$ into Eq. (36), we obtain

$$(B^{-1}\sigma, \tilde{\sigma})_G + (\boldsymbol{u}, \operatorname{div}\tilde{\sigma})_G + (D^{-1}m, \tilde{m})_{\Omega_\gamma} - (w, \nabla\nabla\tilde{m})_{\Omega_\gamma} \geqslant 0.$$

From here, we obtain the inequality

$$\langle [T^\nu(\tilde{m})], w\rangle_{3/2,\Sigma} - \langle \tilde{\sigma}_n, \boldsymbol{u}_n\rangle^{00}_{1/2,\gamma} \geqslant 0$$

and [by virtue of Eq. (54)], the inequality

$$(\varphi, w + \boldsymbol{u}_n \sin\alpha)_\gamma \geqslant 0.$$

As the functions $\varphi \geqslant 0$ are arbitrary, we obtain the inequality $w + \boldsymbol{u}_n \sin\alpha \geqslant 0$ on γ, which supports the validity of Eq. (49).

Finally, we demonstrate that the solution of problem (33)–(36) satisfies the boundary condition

$$\sigma_n(\boldsymbol{un} \sin\alpha + w) = 0 \quad \text{on} \quad \gamma. \tag{55}$$

We choose $(\bar{\sigma}, \bar{m}) = (0, 0)$ and $(\bar{\sigma}, \bar{m}) = 2(\sigma, m)$ as test functions in Eq. (36). Then, we obtain

$$(B^{-1}\sigma, \sigma)_G + (\boldsymbol{u}, \operatorname{div}\sigma)_G + (D^{-1}m, m)_{\Omega_\gamma} - (w, \nabla\nabla m)_{\Omega_\gamma} = 0,$$

and, hence,

$$\langle [T^\nu(m)], w\rangle_{3/2,\Sigma} - \langle \sigma_n, \boldsymbol{u}_n\rangle^{00}_{3/2,\gamma} = 0,$$

which means [with allowance for the last relation in (32)] that Eq. (55) is valid.

1.2. Passage to the Limit in Problem A. For the sake of simplicity, some parameters in problem (1)–(6) are assumed to be equal to unity. In reality, however, the model contains a number of physical and geometrical parameters, and it would be of undoubted interest to consider the dependences on these parameters. Let us pass to the limit with respect to the parameter of plate rigidity. For this purpose, we consider two cases.

1. Instead of Hooke's law $\sigma = B\varepsilon(\boldsymbol{u})$ in Eq. (1), we consider a family of laws

$$\sigma^\beta = \beta^{-1}B\varepsilon(\boldsymbol{u}), \qquad \beta > 0, \tag{56}$$

and pass to the limit as $\beta \to 0$, which corresponds to the case where the lower plate rigidity tends to infinity. The limit problem describes the contact of the upper plate with a thin rigid (non-deformable) obstacle.

2. Instead of Eq. (2), we consider a family of equations

$$\beta^{-1}\Delta^2 w = f, \qquad \beta > 0$$

and pass to the limit as $\beta \to 0$, which describes an increase in the upper plate rigidity to infinity. In the limit problem, the lower plate contacts a rigid obstacle on γ.

First we consider case 1. For an arbitrary fixed $\beta > 0$, we have a unique solution of the problem

$$(\boldsymbol{u}^\beta, w^\beta) \in K; \tag{57}$$

$$(\sigma^\beta(\boldsymbol{u}^\beta), \varepsilon(\bar{\boldsymbol{u}} - \boldsymbol{u}^\beta))_G - (\boldsymbol{g}, \bar{\boldsymbol{u}} - \boldsymbol{u}^\beta)_G + a_\Omega(w^\beta, \bar{w} - w^\beta) - (f, \bar{w} - w^\beta)_\Omega \geqslant 0 \quad \forall\, (\bar{\boldsymbol{u}}, \bar{w}) \in K \tag{58}$$

[$\sigma^\beta(\boldsymbol{u}^\beta) = \sigma^\beta$ were determined in (56)]. Substituting $(\bar{\boldsymbol{u}}, \bar{w}) = (0, 0)$ and $(\bar{\boldsymbol{u}}, \bar{w}) = 2(\boldsymbol{u}^\beta, w^\beta)$ as test functions in (58), we find

$$(\sigma^\beta(\boldsymbol{u}^\beta), \varepsilon(\boldsymbol{u}^\beta))_G - (\boldsymbol{g}, \boldsymbol{u}^\beta)_G + a_\Omega(w^\beta, w^\beta) - (f, w^\beta)_\Omega = 0. \tag{59}$$

Relation (59) yields two estimates

$$\|w^\beta\|_{H_0^2(\Omega)} \leqslant c_1, \qquad \beta^{-1}\|\boldsymbol{u}^\beta\|^2_{H^1_{\gamma_0}(G)} \leqslant c_2$$

with constants c_1 and c_2, which are uniform in terms of β. We can assume that a subsequence with the previous notation $\boldsymbol{u}^\beta$, w^β is convergent as $\beta \to 0$:

$$w^\beta \to w^0 \quad \text{weakly in} \quad H_0^2(\Omega),$$

$$\boldsymbol{u}^\beta \to 0 \quad \text{strongly in} \quad H^1_{\gamma_0}(G).$$

As $\boldsymbol{u}^\beta \boldsymbol{n} \sin\alpha + w^\beta \geqslant 0$ on γ, the limit function w^0 satisfies the inequality

$$w^0 \geqslant 0 \quad \text{on} \quad \gamma. \tag{60}$$

We choose $\bar{w} \in H_0^2(\Omega)$ and $\bar{w} \geqslant 0$ on γ. Then, we have $(0, \bar{w}) \in K$. Substituting the element $(0, \bar{w})$ as a test function in (58), we obtain

$$a_\Omega(w^\beta, \bar{w} - w^\beta) - (f, \bar{w} - w^\beta)_\Omega \geqslant \beta^{-1}(\sigma(\boldsymbol{u}^\beta), \varepsilon(\boldsymbol{u}^\beta))_G - (\boldsymbol{g}, \boldsymbol{u}^\beta)_G.$$

As we have

$$\liminf_{\beta \to 0} \frac{1}{\beta}\,(\sigma(\boldsymbol{u}^\beta), \varepsilon(\boldsymbol{u}^\beta))_G \geqslant 0,$$

the previous inequality implies that

$$w^0 \in M; \tag{61}$$

$$a_\Omega(w^0, \bar{w} - w^0) - (f, \bar{w} - w^0)_\Omega \geqslant 0 \qquad \forall\, \bar{w} \in M. \tag{62}$$

Here

$$M = \{v \in H_0^2(\Omega)\colon v \geqslant 0 \text{ on } \gamma\}.$$

Problem (61), (62) describes the contact between a plate and a thin rigid obstacle aligned along γ. As previously, in problem (61), (62), we can find a full system of boundary conditions satisfied on γ, which have the form

$$[w^0] = [w^0_\nu] = 0, \qquad [m(w^0)] = 0 \quad \text{on} \quad \gamma,$$

$$w^0 \geqslant 0, \qquad [t^\nu(w^0)] \geqslant 0, \qquad [t^\nu(w^0)]w^0 = 0 \quad \text{on} \quad \gamma.$$

Let us consider case 2. For any arbitrary fixed $\beta > 0$, there exists a unique solution of the variational inequality

$$(\boldsymbol{u}^\beta, w^\beta) \in K; \tag{63}$$

$$(\sigma(\boldsymbol{u}^\beta), \varepsilon(\bar{\boldsymbol{u}} - \boldsymbol{u}^\beta))_G - (\boldsymbol{g}, \bar{\boldsymbol{u}} - \boldsymbol{u}^\beta)_G + \beta^{-1}a_\Omega(w^\beta, \bar{w} - w^\beta) - (f, \bar{w} - w^\beta)_\Omega \geqslant 0 \quad \forall\, (\bar{u}, \bar{w}) \in K. \tag{64}$$

Inequality (64) yields the relation

$$(\sigma(\boldsymbol{u}^\beta), \varepsilon(\boldsymbol{u}^\beta))_G - (\boldsymbol{g}, \boldsymbol{u}^\beta)_G + \beta^{-1}a_\Omega(w^\beta, w^\beta) - (f, w^\beta)_\Omega = 0,$$

which ensures the validity of the estimates uniform in β:

$$\beta^{-1}\|w^\beta\|^2_{H_0^2(\Omega)} \leqslant c_3, \qquad \|\boldsymbol{u}^\beta\|_{H^1_{\gamma_0}(G)} \leqslant c_4.$$

Choosing the subsequence, we can assume the following convergence as $\beta \to 0$:

$$w^\beta \to 0 \quad \text{strongly in} \quad H_0^2(\Omega),$$

$$\boldsymbol{u}^\beta \to \boldsymbol{u}^0 \quad \text{weakly in} \quad H^1_{\gamma_0}(G).$$

Obviously, the limit function $\boldsymbol{u}^0$ satisfies the inequality

$$\boldsymbol{u}^0\boldsymbol{n} \geqslant 0 \quad \text{on} \quad \gamma. \tag{65}$$

Choosing the test functions in (64) in the form $(\bar{\boldsymbol{u}}, 0)$, $\bar{\boldsymbol{u}}\boldsymbol{n} \geqslant 0$ on γ, $\bar{\boldsymbol{u}} = (\bar{u}_1, \bar{u}_2) \in H^1_{\gamma_0}(G)$, we obtain

$$(f, w^\beta)_\Omega + (\sigma(\boldsymbol{u}^\beta), \varepsilon(\bar{\boldsymbol{u}} - \boldsymbol{u}^\beta))_G - (\boldsymbol{g}, \bar{\boldsymbol{u}} - \boldsymbol{u}^\beta)_G \geqslant \beta^{-1} a_\Omega(w^\beta, w^\beta). \tag{66}$$

By virtue of the relation

$$\liminf_{\beta \to 0} \frac{1}{\beta}\, a_\Omega(w^\beta, w^\beta) \geqslant 0$$

we can pass to the lower limit in (66), which yields the variational inequality

$$\boldsymbol{u}^0 \in N; \tag{67}$$

$$(\sigma(\boldsymbol{u}^0), \varepsilon(\bar{\boldsymbol{u}} - \boldsymbol{u}^0))_G - (\boldsymbol{g}, \bar{\boldsymbol{u}} - \boldsymbol{u}^0)_G \geqslant 0 \quad \forall\, \bar{\boldsymbol{u}} \in N, \tag{68}$$

where

$$N = \{\boldsymbol{v} = (v_1, v_2) \in H^1_{\gamma_0}(G)\colon \quad \boldsymbol{v}\boldsymbol{n} \geqslant 0 \text{ on } \gamma\}.$$

Note that the limit problem (67), (68) coincides with the classical Signorini problem in the domain G (see [1]).

2. PROBLEM B

2.1. Formulation of Problem B. We consider the case where both plates experience bending only. The problem geometry is the same as that used in Problem A (see Fig. 1). First we consider the differential formulation of the problem. We have to find functions $v(\boldsymbol{x})$ and $w(\boldsymbol{y})$ $[\boldsymbol{x} = (x_1, x_2) \in G$ and $\boldsymbol{y} = (y_1, y_2) \in \Omega_\gamma]$, such that

$$\Delta^2 v = h \quad \text{in} \quad G; \tag{69}$$

$$\Delta^2 w = f \quad \text{in} \quad \Omega_\gamma; \tag{70}$$

$$v = v_n = 0 \quad \text{on} \quad \gamma_0; \tag{71}$$

$$w = w_q = 0 \quad \text{on} \quad \Gamma; \tag{72}$$

$$w - v\cos\alpha \geqslant 0, \qquad t^n(v)(w - v\cos\alpha) = 0 \quad \text{on} \quad \gamma; \tag{73}$$

$$[w] = [w_\nu] = 0, \qquad [m(w)] = 0 \quad \text{on} \quad \gamma; \tag{74}$$

$$t^n(v) \leqslant 0, \qquad m(v) = 0, \qquad [t^\nu(w)]\cos\alpha = -t^n(v) \quad \text{on} \quad \gamma. \tag{75}$$

Here

$$v_n = \frac{\partial v}{\partial n}, \quad m(w) = \varkappa_1 \Delta w + (1 - \varkappa_1)\,\frac{\partial^2 w}{\partial \nu^2}, \quad t^\nu(w) = \frac{\partial}{\partial \nu}\Big(\Delta w + (1 - \varkappa_1)\,\frac{\partial^2 w}{\partial s^2}\Big).$$

The values of $m(v)$ and $t^n(v)$ are determined similar to $m(w)$ and $t^\nu(w)$:

$$m(v) = \varkappa_2 \Delta v + (1 - \varkappa_2)\,\frac{\partial^2 v}{\partial n^2}, \qquad t^n(v) = \frac{\partial}{\partial n}\Big(\Delta v + (1 - \varkappa_2)\,\frac{\partial^2 v}{\partial \tau^2}\Big).$$

Here $\varkappa_2$ is Poisson's ratio for the second plate; $(\tau_1, \tau_2) = (-n_2, n_1)$. The function v describes the vertical (normal) displacements of the lower plate, and the functions $h \in L^2(G)$ and $f \in L^2(\Omega)$ are assumed to be given.

Below we prove the solvability of problem (69)–(75). For this purpose, we use the variational formulation of the problem.

We introduce the Sobolev space

$$H^2_{\gamma_0}(G) = \{v \in H^2(G)\colon v = v_n = 0 \text{ on } \gamma_0\}$$

and a set of admissible displacements

$$S = \{(v,w)\colon v \in H^2_{\gamma_0}(G),\ w \in H^2_0(\Omega),\ w - v\cos\alpha \geqslant 0 \quad \text{on} \quad \gamma\}.$$

For the lower plate, we use the following bilinear form:

$$a_G(v,\bar{v}) = \int\limits_G (v_{,11}\bar{v}_{,11} + v_{,22}\bar{v}_{,22} + \varkappa_2(v_{,11}\bar{v}_{,22} + v_{,22}\bar{v}_{,11}) + 2(1-\varkappa_2)v_{,12}\bar{v}_{,12}).$$

We consider the energy functional

$$\Pi(v,w) = a_G(v,v)/2 + a_\Omega(w,w)/2 - (h,v)_G - (f,w)_\Omega$$

and the minimization problem

$$\inf_{(v,w)\in S} \Pi(v,w), \tag{76}$$

which is equivalent to the variational inequality

$$(v,w) \in S; \tag{77}$$

$$a_G(v,\bar{v}-v) + a_\Omega(w,\bar{w}-w) - (h,\bar{v}-v)_G - (f,\bar{w}-w)_\Omega \geqslant 0 \quad \forall\,(\bar{v},\bar{w}) \in S. \tag{78}$$

The set S is weakly closed in the space $H^2_{\gamma_0}(G) \times H^2_0(\Omega)$, and the functional Π is coercive and weakly lower semi-continuous on this space. Hence, the problem of minimization (76) has a solution satisfying the variational inequality (77), (78). This solution is unique.

From (77) and (78), we derive the equations and boundary conditions (69)–(75) and find in which sense conditions (73)–(75) are satisfied.

Note that Eqs. (69) and (70) follow from (78) and are satisfied in the sense of distributions. Indeed, we can substitute the test functions $(\bar{v},\bar{w}) = (v \pm \varphi, w \pm \psi)$, where $\varphi \in C_0^\infty(G)$ and $\psi \in C_0^\infty(\Omega_\gamma)$, in Eq. (78), which implies the validity of Eqs. (69) and (70).

As in Sec. 1, we consider the extension of the curve γ up to a closed curve Σ of class $C^{1,1}$, such that $\Sigma \subset \Omega$. We assume that the vector $\boldsymbol{\nu} = (\nu_1,\nu_2)$ is defined on the entire curve Σ, being an outward vector to Ω_1. We choose $(\bar{v},\bar{w}) = (v, w+\psi)$ as test functions in (78), with $\psi \geqslant 0$ on γ, $\psi \in H^2_0(\Omega)$. From here we obtain the inequality

$$a_\Omega(w,\psi) - (f,\psi)_\Omega \geqslant 0. \tag{79}$$

Green's formula (18) in combination with Eq. (70) allows us to obtain the following inequality from Eq. (79):

$$-\langle [m(w)], \psi_\nu\rangle_{1/2,\Sigma} + \langle [t^\nu(w)], \psi\rangle_{3/2,\Sigma} \geqslant 0.$$

As ψ_ν are arbitrary on Σ, we obtain

$$[m(w)] = 0 \quad \text{in the sense} \quad H^{-1/2}(\Sigma); \tag{80}$$

$$\langle [t^\nu(w)], \psi\rangle_{3/2,\Sigma} \geqslant 0 \quad \forall\,\psi \in H^2_0(\Omega), \qquad \psi \geqslant 0 \quad \text{on} \quad \gamma. \tag{81}$$

We choose $(\bar{v},\bar{w}) = (v+\varphi, w)$ as test functions in (78), with $\varphi \in H^2_{\gamma_0}(G)$, $\varphi \leqslant 0$ on γ. Then, we obtain the inequality

$$a_G(v,\varphi) - (h,\varphi)_G \geqslant 0. \tag{82}$$

With allowance for the equilibrium equation (69) and a formula of the form (18), we obtain the following relation for the domain G:

$$-\langle m(v), \varphi_n\rangle_{1/2,\partial G} + \langle t^n(v), \varphi\rangle_{3/2,\partial G} \geqslant 0. \tag{83}$$

Note that $\boldsymbol{n}$ is the inward normal to ∂G; hence, we have the plus sign at the second term in inequality (83) and the minus sign at the first term. In our case, we have $\varphi = \varphi_n = 0$ on $\partial G \setminus \gamma$. Hence, we obtain $\varphi \in H^{3/2}_{00}(\gamma)$ and $\varphi_n \in H^{1/2}_{00}(\gamma)$. Therefore, we can write inequality (83) in the form

$$-\langle m(v), \varphi_n\rangle^{00}_{1/2,\gamma} + \langle t^n(v), \varphi\rangle^{00}_{3/2,\gamma} \geqslant 0 \quad \forall\,\varphi \in H^{3/2}_{00}(\gamma), \qquad \varphi \leqslant 0 \quad \text{on} \quad \gamma,$$

whence there follow two conditions

$$m(v) = 0 \quad \text{in the sense} \quad H_{00}^{-1/2}(\gamma); \tag{84}$$

$$t^n(v) \leqslant 0 \quad \text{in the sense} \quad H_{00}^{-3/2}(\gamma). \tag{85}$$

Similarly, assuming that $\psi = 0$ outside γ, i.e., $\psi = 0$ on $\Sigma \setminus \gamma$, we obtain the following relation from Eq. (81):

$$[t^\nu(w)] \geqslant 0 \quad \text{in the sense} \quad H_{00}^{-3/2}(\gamma). \tag{86}$$

Substituting $(\bar{v}, \bar{w}) = (v, w) \pm (\varphi, \psi)$ as test functions in (78), where $(\varphi, \psi) \in S$ and $-\varphi \cos\alpha + \psi = 0$ on γ, we obtain

$$a_G(v, \varphi) + a_\Omega(w, \psi) - (h, \varphi)_G - (f, \psi)_\Omega = 0.$$

By virtue of Eqs. (69), (70), (80), and (84), this relation yields

$$\langle t^n(v), \varphi\rangle_{3/2,\partial G} + \langle [t^\nu(w)], \psi\rangle_{3/2,\Sigma} = 0.$$

Assuming that $\psi = 0$ on Σ outside γ, we find

$$[t^\nu(w)] \cos\alpha = -t^n(v) \quad \text{in the sense} \quad H_{00}^{-3/2}(\gamma). \tag{87}$$

We choose $(\bar{v}, \bar{w}) = (v, w) + (\varphi, \psi)$ in (78), with $(\varphi, \psi) \in S$. As a result, we obtain

$$a_G(v, \varphi) + a_\Omega(w, \psi) - (h, \varphi)_G - (f, \psi)_\Omega \geqslant 0 \qquad \forall\, (\varphi, \psi) \in S.$$

Hence,

$$\langle t^n(v), \varphi\rangle_{3/2,\gamma}^{00} + \langle [t^\nu(w)], \psi\rangle_{3/2,\Sigma} \geqslant 0 \quad \forall\, (\varphi, \psi) \in S. \tag{88}$$

The resultant inequality yields the exact formulation of the boundary conditions

$$t^n(v) \leqslant 0, \qquad [t^\nu(w)] \cos\alpha = -t^n(v) \quad \text{on} \quad \gamma.$$

Note that Eqs. (86) and (87) can also be derived from Eq. (88).

Substituting $(\bar{v}, \bar{w}) = (0, 0)$ and $(\bar{v}, \bar{w}) = 2(v, w)$ as test functions into (78), we obtain

$$a_G(v, v) + a_\Omega(w, w) - (h, v)_G - (f, w)_\Omega = 0.$$

Hence,

$$\langle t^n(v), v\rangle_{3/2,\gamma}^{00} + \langle [t^\nu(w)], w\rangle_{3/2,\Sigma} = 0. \tag{89}$$

The resultant relation is the exact formulation of the boundary conditions [see Eqs. (73) and (75)]

$$[t^\nu(w)] \cos\alpha = -t^n(v), \qquad t^n(v)(w - v\cos\alpha) = 0 \quad \text{on} \quad \gamma.$$

To conclude, we should note that the variational inequality (77) and (78) can be derived from (69)–(75). Thus, the system of the boundary conditions (73)–(75) is complete on γ.

2.2. Passing to the Limit in Problem B. We study the limit transition with a vanishing parameter characterizing plate rigidity. For this purpose, instead of Eq. (69), we consider a family of equations depending on the parameter β:

$$\beta^{-1}\Delta^2 v = h, \qquad \beta > 0.$$

The problem reduces to justification of the limit transition for $\beta \to 0$, which corresponds to rigidity of the lower plate tending to infinity.

We consider the variational formulation of the problem for each fixed value $\beta > 0$. There exists a unique solution v^β, w^β of the following problem:

$$(v^\beta, w^\beta) \in S; \tag{90}$$

$$\beta^{-1} a_G(v^\beta, \bar{v} - v^\beta) + a_\Omega(w^\beta, \bar{w} - w^\beta) - (h, \bar{v} - v^\beta)_G - (f, \bar{w} - w^\beta)_\Omega \geqslant 0 \quad \forall\, (\bar{v}, \bar{w}) \in S. \tag{91}$$

Substituting $(\bar{v}, \bar{w}) = (0, 0)$, $(\bar{v}, \bar{w}) = 2(v^\beta, w^\beta)$ into (91), we obtain the equality

$$\beta^{-1} a_G(v^\beta, v^\beta) + a_\Omega(w^\beta, w^\beta) - (h, v^\beta)_G - (f, w^\beta)_\Omega = 0, \tag{92}$$

which yields two estimates

$$\|w^\beta\|^2_{H^2_0(\Omega)} \leqslant c_5, \qquad \beta^{-1}\|v^\beta\|^2_{H^2_{\gamma_0}(G)} \leqslant c_6 \tag{93}$$

with constant c_5 and c_6 uniform in β. We assume that the subsequence with the previous notation v^β, w^β possesses the following property as $\beta \to 0$:

$$w^\beta \to w^0 \quad \text{weakly in} \quad H^2_0(\Omega), \tag{94}$$

$$v^\beta \to 0 \quad \text{strongly in} \quad H^2_{\gamma_0}(G). \tag{95}$$

As $-v^\beta \cos\alpha + w^\beta \geqslant 0$ on γ, the limit function w^0 satisfies the inequality

$$w^0 \geqslant 0 \quad \text{on} \quad \gamma. \tag{96}$$

We choose the test functions in inequality (91) in the form $(0, \bar{w})$, with $\bar{w} \in H^2_0(\Omega)$, $\bar{w} \geqslant 0$ on γ. Then, we obtain

$$a_\Omega(w^\beta, \bar{w} - w^\beta) \geqslant \beta^{-1} a_G(v^\beta, v^\beta) + (h, v^\beta)_G - (f, \bar{w} - w^\beta)_\Omega.$$

As we have

$$\liminf_{\beta\to 0} \frac{1}{\beta}\, a_G(v^\beta, v^\beta) \geqslant 0,$$

the above-derived inequality yields

$$w^0 \in M; \tag{97}$$

$$a_\Omega(w^0, \bar{w} - w^0) - (f, \bar{w} - w^0)_\Omega \geqslant 0 \qquad \forall\, \bar{v} \in M, \tag{98}$$

where

$$M = \{u \in H^2_0(\Omega)\colon u \geqslant 0 \text{ on } \gamma\}.$$

Thus, the limit problem (97), (98) describes the contact of the upper plate with an infinitely thin rigid obstacle aligned along γ.

Actually, we can demonstrate that the convergence is stronger than that in (94) and (95). To prove this statement, we should recall that Eq. (92) implies that

$$\begin{aligned}\limsup_{\beta\to 0} \frac{1}{\beta}\, a_G(v^\beta, v^\beta) &= \limsup_{\beta\to 0}\{-a_\Omega(w^\beta, w^\beta) + (h, v^\beta)_\Omega + (f, w^\beta)_\Omega\}\\ &\leqslant \limsup_{\beta\to 0}\{-a_\Omega(w^\beta, w^\beta)\} + \limsup_{\beta\to 0}(h, v^\beta)_G + \limsup_{\beta\to 0}(f, w^\beta)_\Omega\\ &\leqslant -a_\Omega(w^0, w^0) + (f, w^0)_\Omega.\end{aligned}$$

Simultaneously, it follows from Eq. (98) that

$$a_\Omega(w^0, w^0) = (f, w^0)_\Omega. \tag{99}$$

Thus, the above-described reasoning leads us to the relations

$$0 \leqslant \liminf_{\beta\to 0} \frac{1}{\beta}\, a_G(v^\beta, v^\beta) \leqslant \limsup_{\beta\to 0} \frac{1}{\beta}\, a_G(v^\beta, v^\beta) \leqslant 0,$$

which prove the following convergence as $\beta \to 0$:

$$\beta^{-1} a_G(v^\beta, v^\beta) \to 0. \tag{100}$$

From here, in addition to (95), we obtain the property

$$v^\beta/\sqrt{\beta} \to 0 \quad \text{strongly in} \quad H^2_{\gamma_0}(G).$$

After that, in addition to convergence (94), we can prove the convergence

$$w^\beta \to w^0 \quad \text{strongly in} \quad H^2_0(\Omega). \tag{101}$$

Indeed, as the weak convergence of the sequence w^β to w^0 is verified, it suffices to show that the following statement is valid as $\beta \to 0$:

$$a_\Omega(w^\beta, w^\beta) \to a_\Omega(w^0, w^0). \tag{102}$$

From Eq. (92), we obtain

$$a_\Omega(w^\beta, w^\beta) = -a_G(v^\beta, v^\beta)/\beta + (h, v^\beta)_G + (f, w^\beta)_\Omega.$$

By virtue of Eqs. (95) and (100), the expression in the right side of this inequality has a limit equal to $(f, w^0)_\Omega$, i.e., as $\beta \to 0$, we have

$$\lim a_\Omega(w^\beta, w^\beta) = (f, w^0)_\Omega.$$

With allowance for Eq. (99), we obtain Eq. (102), which proves convergence (101).

This work was supported by the Russian Foundation for Basic Research (Grant No. 06-01-00209).

REFERENCES

1. G. Fichera, "Boundary value problems of elasticity with unilateral constraints," in: *Handbuch der Physik*, Band 6a/2, Springer-Verlag, Berlin–Heidelberg–New York (1972).
2. A. M. Khludnev and J. Sokolowski, *Modelling and Control in Solid Mechanics*, Birkhäuser, Basel–Boston–Berlin (1997).
3. A. M. Khludnev, "Extreme shapes of cuts in a plate," *Izv. Ross. Akad. Nauk, Mekh. Tverd. Tela*, No. 1, 170–176 (1992).
4. A. M. Khludnev and V. A. Kovtunenko, *Analysis of Cracks in Solids*, WIT Press, Southampton–Boston (2000).
5. A. M. Khludnev, "Theory of cracks with a possible contact of the crack edges," *Usp. Mekh.*, **3**, No. 4, 41–82 (2005).
6. A. M. Khludnev, K.-H. Hoffmann, and N. D. Botkin, "Variational problem of the contact of elastic objects of different dimensions," *Sib. Mat. Zh.*, **47**, No. 3, 707–717 (2006).
7. L. A. Caffarelli and A. Friedman, "The obstacle problem for the biharmonic operator," *Ann. Scuola Norm. Sup. Pisa, Ser.* IV, **6**, No. 1, 151–184 (1979).
8. L. A. Caffarelli, A. Friedman, and A. Torelli, "The two-obstacle problem for the biharmonic operator," *Pacific J. Math.*, **103**, No. 3, 325–335 (1982).
9. G. Dal Maso and G. Paderni, "Variational inequalities for the biharmonic operator with varying obstacles," *Ann. Mat. Pura Appl.*, **153**, 203–227 (1988).
10. B. Schild, "On the coincidence set in biharmonic variational inequalities with thin obstacles," *Ann. Scuola Norm. Sup. Pisa, Cl. Sci. Ser. IV*, **13**, No. 4, 559–616 (1986).
11. P. Grisvard, *Elliptic Problems in Nonsmooth Domains*, Pitman, Boston–London–Melbourne (1985).
12. R. Temam, *Problemes Mathematiques en Plasticite*, Gauthier-Villars, Paris (1983).